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Translated by L.K.

PMM U.S.S.R.,Vol.51,No.4,pp.462-467,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON TURBULENT BOUNDARY LAYER STRUCTURE*

V.V. SYCHEV and VIK. V. SYCHEV

Flow in the turbulent boundary layer (BL) with Reynolds number $R\to\infty$ is studied by a joining asymptotic expansion method. A three-layered asymptotic BL structure is set up, which includes, besides the viscous boundary region and the velocity defect region, an intermediate region in which a balance of inertia forces, and pressure and turbulent friction forces takes place and which is responsible for flow separation under the influence of a disadvantageous pressure gradient.

A study of turbulent BL structure, based on the asymptotic analysis of an open set of Reynolds equations, has been the subject of a number of investigations. Early papers /1, 2/ essentially contain the known elements of this type of analysis. The paper by Yajnik /3/ was the first attempt at a systematic approach to the problem of constructing joined asymptotic expansions for averaged flow functions in a turbulent BL as $R\to\infty$. Further developments were made in /4-6/. In all these studies the structure of the turbulent BL, either with or without a pressure gradient, was established as a double layer: an inner (boundary) region and an outer region. In the first of these, flow is defined by a known Prandtl wall law which states that the sum of friction stresses caused by viscous action and turbulent pulses of velocity remains invariant across the whole zone. Flow in the outer region is described by the Kármán velocity defect law and represents a slightly perturbed potential flow close to the solid surface.

The possibility of a formal joining of the solutions for these regions is often seen as proof of the existence of an overlap region between them and a logarithmic velocity profile. The joining conditions also make it possible to find that the relative thickness of the velocity defect region is of the order of $1/\ln R$, and that the thickness of the layer at the wall is of the order of $\ln R/R$ /3/.

A more detailed examination of flow in the boundary region carried out below, however, shows that the two-layered flow diagram does not take place in reality. This diagram does not contain a region where the internal friction forces, the pressure gradient and inertia forces have the same order of magnitude as $R\to\infty$, i.e. just that region which, according to the Prandtl definition, is itself a BL. This, in particular, excludes the possibility of explaining the flow separation under a disadvantageous pressure gradient. Indeed, the flow in the velocity defect region to a first approximation is not susceptible to the action of friction forces and the flow in the wall-law region is not subject to the pressure gradient (since, for this to be so, the latter must have an unpracticably large values of the order of $R/\ln^3 R$).

In /7/, based on experimental observations, a law of the wake was introduced into the consideration, successfully linking the laws of the wall and the velocity defect. By this law the velocity profile in the BL is essentially dependent on the pressure gradient and changes so that, as it approaches the separation point, it adopts the shape of a profile in the wake. The law of the wake can therefore be considered as proof that the BL structure is not two-layered, i.e. the overlap region of the wall and velocity defect law does not exist in reality (even for flow without a pressure gradient) and, consequently, it is essential to

*Prikl.Matem.Mekhan.,51,4,593-599,1987

introduce at least one more intermediate zone.

The construction of an asymptotic turbulent BL theory by introducing such a zone was attempted in /8/. The use of a logarithmic law for the behaviour of the velocity defect as a joining condition, in essence, signified a return to the scheme in /3/, since the average longitudinal flow velocity in the intermediate zone was also, in its main term, equal to the potential flow velocity at the outer BL boundary.

1. Let us being the flow analysis in the boundary layer with an examination of the boundary region. In accordance with the law of the wall the defining parameter here is the dynamic velocity $u^* = \sqrt{\tau_w/\rho}$, where τ_w is the friction at the wall and ρ is the density of the environment. The ratio of the characteristic value of this velocity u_0^* to the characteristic velocity of the external inviscid flow U_∞, as usual, we will consider as the small parameter of problem $\varepsilon = u_0^*/U_\infty$. Taking the dimenionless averaged values of the longitudinal component of the velocity vector and friction in the boundary regions to be values of the order of ε and ε^2 respectively, it is possible to determine the orders of magnitude of various terms of Reynolds equations.

Let u and v be dimensionless averaged values of the velocity vector components along the Ox and Oy axes of a rectangular Cartesian system of coordinate connected to the plane smooth surface of solids $y = 0$, and let p and τ_{ij} be dimensionless values of the average pressure drop and turbulent stresses, L is the characteristic size of solid chosen as the unit of length measurement. We will write the Reynolds equations for planar flow in the form

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{1.1}$$
$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \quad R = \frac{U_\infty L}{\nu}$$

The layer at the wall is characterized by the fact that viscous and turbulent friction stresses possess an identical order of magnitude here, i.e. $R^{-1}\partial u / \partial y \sim \tau_{xy}$. Using the estimates stated above for the values of the longitudinal component of the velocity vector u and the friction, we find that the thickness of the viscous layer at the wall is

$$\delta = O\left(1/(\varepsilon R)\right) \tag{1.2}$$

and the corresponding terms in the first equation of the angular momentum (1.1)

$$R^{-1}\partial^2 u/\partial y^2 \sim \partial\tau_{xy} / \partial y \sim \varepsilon^3 R$$

We will assume that the dimensionless longitudinal pressure gradient in the boundary layer is a quantity of the order of unity; the contribution of inertial terms, for example $u\partial u / \partial x$, based on the estimates presented above will be of the order of ε^2. Therefore, using the initial Eqs.(1.1) it is possible to write the following asymptotic expansions of the unknown functions for flow in the region at the wall of the boundary layer as

$$u = \varepsilon\left(u_1^+ + \frac{1}{\varepsilon^3 R}u_2^+ + \frac{\ln\varepsilon}{\varepsilon R}u_3^+ + \frac{1}{\varepsilon R}u_4^+ + \ldots\right) \tag{1.3}$$
$$v = \frac{1}{R}\left(v_1^+ + \frac{1}{\varepsilon^3 R}v_2^+ + \frac{\ln\varepsilon}{\varepsilon R}v_3^+ + \frac{1}{\varepsilon R}v_4^+ + \ldots\right)$$
$$p = p_1^+ + \varepsilon^2\ln\varepsilon\, p_2^+ + \varepsilon^2 p_3^+ + \ldots$$
$$\tau_{xy} = \varepsilon^2\left(\tau_1^+ + \frac{1}{\varepsilon^3 R}\tau_2^+ + \frac{\ln\varepsilon}{\varepsilon R}\tau_3^+ + \frac{1}{\varepsilon R}\tau_4^+ + \ldots\right)$$
$$\tau_{xx} = \varepsilon^2\pi_1^+ + \ldots, \quad \tau_{yy} = \varepsilon^2\sigma_1^+ + \ldots$$

The independent variables of the order of unity here are

$$x,\ y^+ = \varepsilon R y \tag{1.4}$$

The apperance in (1.3) of terms containing $\ln\varepsilon$ is caused by pressure changes due to the displacing action of the boundary layer, as will be shown below. (We recall that for simplicity the case of flow over a plane surface is examined; in the case of flow around a curvilinear wall the term of order ε will still appear in the expansion for pressure.)

As a result of substituting (1.3) and (1.4) into (1.1) we arrive at the following system of relations:

$$\frac{\partial^2 u_1^+}{\partial y^{+2}} + \frac{\partial \tau_1^+}{\partial y^+} = 0, \quad \frac{\partial^2 u_2^+}{\partial y^{+2}} + \frac{\partial \tau_2^+}{\partial y^+} = \frac{\partial p_1^+}{\partial x}, \quad \frac{\partial p_1^+}{\partial y^+} = 0 \tag{1.5}$$
$$\frac{\partial^2 u_3^+}{\partial y^{+2}} + \frac{\partial \tau_3^+}{\partial y^+} = \frac{\partial p_2^+}{\partial x}, \quad \frac{\partial p_2^+}{\partial y^+} = 0$$

$$\frac{\partial^2 u_4^+}{\partial y^{+2}} + \frac{\partial \tau_4^+}{\partial y^+} + \frac{\partial \pi_1^+}{\partial x} = \frac{\partial p_3^+}{\partial x} + u_1^+ \frac{\partial u_1^+}{\partial x} + v_1^+ \frac{\partial u_1^+}{\partial y^+}$$

$$\frac{\partial p_3^+}{\partial y^+} = \frac{\partial \sigma_1^+}{\partial y^+}; \quad \frac{\partial u_k^+}{\partial x} + \frac{\partial v_k^+}{\partial y^+} = 0, \quad k = 1, 2, 3, 4$$

On the surface of the solid $y^+ = 0$ the functions must satisfy the adhesion condition.

From this system it is possible to determine the behaviour of the functions occurring in it as $y^+ \to \infty$. Thus, for functions of the first approximation as $y^+ \to \infty$ as usual, let us assume

$$u_1^+ = h_1(x) \ln y^+ + f_1(x) + \ldots, \tau_1^+ = g_1(x) - h_1(x) / y^+ + \ldots \tag{1.6}$$

Here $g_1(x)$ is the dimensionless friction on the wall, and $h_1(x)$ and $f_1(x)$ are certain derived functions. In the second of these expansions we used the assumption that the influence of viscosity on the magnitude of the internal friction in the inner part of the subboundary layer decreases in inverse proportion to y^+. This assumption corresponds to the logarithmic law, due to which, as is well-known, the value of the longitudinal component u outside the viscous sublayer becomes a value of the order of unity and, besides this, here the internal limit of this function is non-zero.

Considering Eqs.(1.5) for the second and third approximations it is possible to establish that

$$\tau_2^+ = p_1^{+\prime}(x) y^+ + \ldots, \tau_3^+ = p_2^{+\prime}(x) y^+ + \ldots (y^+ \to \infty) \tag{1.7}$$

and, as a result of substituting (1.6) into the right-hand side of the equation for functions of the fourth approximation in (1.5) we obtain

$$\tau_4^+ = h_1(x) h_1'(x) y^+ \ln^2 y^+ + O(y^+ \ln y^+) (y^+ \to \infty) \tag{1.8}$$

Based on the asymptotic representations (1.6), (1.7) for τ_1^+ and τ_2^+ as $y^+ \to \infty$ we find that the corresponding terms of expansion (1.3) for τ_{xy} become of the same order for $y^+ = O(\varepsilon^3 R)$. The fourth term of this expansion here will be of the same order of smallness if $\varepsilon \ln(\varepsilon^3 R) = O(1)$. We note that the value of the velocity component u, in accordance with (1.3), and (1.6), then becomes a value of the order of unity where $y^+ = O(\varepsilon^3 R)$. Therefore, it is possible to set

$$\varepsilon = 1 / \ln(\varepsilon^3 R) = 1 / \ln R + \ldots \tag{1.9}$$

without loss of generality, which corresponds to the result obtained in /3/, mentioned above.

Thus we obtain that expansion (1.3) for τ_{xy} ceases to be correct in a region with relative transverse dimensions $\varepsilon^3 R\delta$ or ε^2 (in accordance with (1.2)).

Now let us make the assumption that the regions of inapplicability of the expansions for u and τ_{xy} coincide, i.e. the terms of expansion (1.3) for u are equal in order of magnitude also where $y^+ = O(\varepsilon^3 R)$ or $y = O(\varepsilon^2)$. Essentially this assumption implicitly postulates the presence of a certain additional dependence between the flow functions, apart from the system of Eqs.(1.5). Such a dependence, however, is much less restrictive than any condition of the closure of this set (since it can be non-local).

The main terms of the asymptotic representations u_2^+, u_3^+ and u_4^+ as $y^+ \to \infty$ must take the form

$$u_2^+ = h_2(x) y^+ \ln y^+ + \ldots, \quad u_3^+ = h_3(x) y^+ \ln y^+ + \ldots \tag{1.10}$$
$$u_4^+ = h_4(x) y^+ \ln^3 y^+ + \ldots$$

where $h_2(x)$, $h_3(x)$ and $h_4(x)$ are arbitrary functions. Since the asymptotic representations (1.10) and (1.7), (1.8) are in line with each other and on the basis of Eqs.(1.5), we can determine the following terms in expressions (1.7), which will be $-h_2(x) \ln y^+$ and $-h_3(x) \ln y^+$ respectively.

2. We consider the flow region with independent variables

$$x, Y = y / \varepsilon^2 = y^+ / (\varepsilon^3 R) \tag{2.1}$$

Based on asymptotic expansions (1.3) and expressions (1.6)-(1.10) and also using the condition for the solutions to match for regions $y^+ = O(1)$ and $Y = O(1)$, we can write the expansions for flow parameters in region $Y = O(1)$ in the form

$$\begin{aligned} u &= u_1 + \varepsilon u_2 + O(\varepsilon^2 \ln \varepsilon), \ v = \varepsilon^2 v_1 + \varepsilon^3 v_2 + \\ &\quad O(\varepsilon^4 \ln \varepsilon), \ p = p_1 + \varepsilon^2 \ln \varepsilon p_2 + O(\varepsilon^2) \\ \tau_{xy} &= \varepsilon^2 \tau_1 + \varepsilon^3 \tau_2 + O(\varepsilon^4 \ln \varepsilon) \\ \tau_{xx} &= \varepsilon^2 \pi_1 + O(\varepsilon^3), \ \tau_{yy} = \varepsilon^2 \sigma_1 + O(\varepsilon^3) \end{aligned} \tag{2.2}$$

Substituting these expansions into the Reynolds Eqs.(1.1), rewritten in variables (2.1), we obtain a set of boundary layer equations for the main terms of (2.2) in the form

$$u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial Y} + \frac{\partial p_1}{\partial x} = \frac{\partial \tau_1}{\partial Y} \tag{2.3}$$

$$\frac{\partial p_1}{\partial Y}=0, \quad \frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial Y}=0$$

The boundary conditions which must be satisfied by functions (2.3) for $Y=0$ are defined by the conditions for matching with the solution for the viscous subboundary layer. Based on (1.3), (1.6)-(1.10) and (2.1) we find that

$$p_1(x)=p_1^+(x),\ v_1(x,0)=0;\ u_1=h_1(x)+O(Y)$$
$$\tau_1=g_1(x)+[p_1'(x)+h_1(x)h_1'(x)]Y+\ldots\ (Y\to 0)$$

For the functions of the subsequent approximation in expansions (2.2) from matching with the solution in the region $y^+=O(1)$ it follows that

$$p_2(x)=p_2^+(x),\ v_2(x,0)=0$$
$$u_2=h_1(x)\ln Y+f_1(x)+O(Y\ln Y)$$
$$\tau_2=2h_1(x)h_1'(x)Y\ln Y+O(Y)\quad (Y\to 0)$$

3. The boundary conditions as $Y\to\infty$ are defined by the conditions for matching with the solution for the velocity defect region. Therefore it is essential to examine flow in this region. The longitudinal component of the velocity vector u is equal here to the first approximation of the velocity $U_e(x)$ of the outer potential flow and the turbulent friction only affects its value in the second approximation. The relative thickness of this region is, as is well-known a value of the order of ε /3/. Therefore, it is necessary to introduce

$$x,\ y^*=y/\varepsilon=\varepsilon Y \tag{3.1}$$

as independent variables of the order of unity.

The averaged flow functions are presented in the form of expansions

$$\begin{aligned} u&=U_e+\varepsilon U_1+\varepsilon^2\ln\varepsilon\, U_2+\varepsilon^2U_3+\ldots\\ v&=\varepsilon V_1+\varepsilon^2\ln\varepsilon V_2+\varepsilon^2V_3+\ldots\\ p&=P_1+\varepsilon^2\ln\varepsilon P_2+\varepsilon^2P_3+\ldots\\ \tau_{xy}&=\varepsilon^2T_1+\varepsilon^3\ln\varepsilon T_2+\varepsilon^3T_3+\ldots\\ \tau_{xx}&=\varepsilon^2\Pi_1+\ldots,\ \tau_{yy}=\varepsilon^2\Sigma_1+\ldots \end{aligned} \tag{3.2}$$

Substituting these expansions together with (3.1) into Eqs.(1.1) we obtain

$$\begin{aligned} &P_1'=p_e'(x)=-U_e(x)U_e'(x),\quad V_1=-U_e'(x)y^*,\quad V_2=H_2(x)\\ &U_e'U_1+U_e\frac{\partial U_1}{\partial x}-U_e'y^*\frac{\partial U_1}{\partial y^*}=\frac{\partial T_1}{\partial y^*},\quad \frac{\partial U_1}{\partial x}+\frac{\partial V_3}{\partial y^*}=0\\ &U_e'U_2+U_e\frac{\partial U_2}{\partial x}-U_e'y^*\frac{\partial U_2}{\partial y^*}+H_2\frac{\partial U_1}{\partial y^*}+\frac{\partial P_2}{\partial x}=\frac{\partial T_2}{\partial y^*}\\ &U_e'U_3+U_e\frac{\partial U_3}{\partial x}-U_e'y^*\frac{\partial U_3}{\partial y^*}+U_1\frac{\partial U_1}{\partial x}+V_3\frac{\partial U_1}{\partial y^*}+\frac{\partial P_3}{\partial x}=\\ &\quad\frac{\partial \Pi_1}{\partial x}+\frac{\partial T_3}{\partial y^*},\quad \frac{\partial P_2}{\partial y^*}=0\\ &(U_e'^2-U_eU_e'')y^*+\frac{\partial P_3}{\partial y^*}=\frac{\partial\Sigma_1}{\partial y^*} \end{aligned} \tag{3.3}$$

In the region where $y=O(\varepsilon^2)$ a non-linearity of the initial equations appears and so it must be assumed that expansions (3.2), like expansions (1.3), become invalid here. This means that, as $y^*\to 0$

$$U_1=F_1(x^*)/y^*+\ldots \tag{3.4}$$

We substitute this expression into (3.3) and require that

$$T_1=G_1(x)+\ldots,\quad y^*\to 0 \tag{3.5}$$

i.e. the matching condition for the main term of the expansion of the function τ_{xy} as $Y\to\infty$ and $y^*\to 0$ has been satisfied. Then for the function $F_1(x)$ we obtain the equation $U_eF_1'+2U_e'F_1=0$ and by integrating we find that

$$F_1=c_1/U_e^2(x) \tag{3.6}$$

where c is an arbitrary constant. From the equation of continuity in (3.3) it follows that

$$V_3=-F_1'(x)\ln y^*+H_3(x)+\ldots,\quad y^*\to 0 \tag{3.7}$$

To carry out the matching of expansions (2.2) and (3.2) for the function v we rewrite (3.7) in terms of the variable Y:

$$V_3=-F_1'(x)\ln\varepsilon-F_1'(x)\ln Y+\ldots$$

and since the term of order $\varepsilon^2\ln\varepsilon$ in expansion (2.2) for v is not present, it is essential to set

$$H_2(x) = F_1'(x) = -2c_1U_e'(x) / U_e^3(x) \tag{3.8}$$

The asymptotic matching for the following terms of expansions (3.2), achieved using expressions (3.4), (3.6)-(3.8) and Eqs.(3.3), makes it possible to establish that as $y^* \to 0$

$$U_2 = \frac{F_2(x)}{y^{*2}} + \ldots, \quad U_3 = -\frac{F_2(x)}{y^{*2}} \ln y^* + \frac{F_3(x)}{y^{*2}} + \ldots$$
$$\mathrm{T}_2 = \frac{G_2(x)}{y^*} + \ldots, \quad \mathrm{T}_3 = -\frac{G_2(x)}{y^*} \ln y^* + \frac{G_3(x)}{y^*} + \ldots$$
$$G_2 = F_1F_1' - 3U_e'F_2 - U_eF_2'$$
$$G_3 = -G_2 - F_2U_e' - F_1(F_1' - H_3) - 3F_3U_e' - F_3'U_e$$

Here the functions $F_2(x), F_3(x), H_3(x)$ and also $G_1(x)$ in (3.5) remain arbitrary.

For the functions u_1, v_1, τ_1, in expansions (2.2) we also obtain the following representations as $Y_1 \to \infty$:

$$u_1 = U_e(x) + \frac{F_1(x)}{Y} - \frac{F_2(x)}{Y^2} \ln Y + \frac{F_3(x)}{Y^2} + \ldots$$
$$v_1 = -U_e'(x)Y - F_1'(x) \ln Y + H_3(x) - \frac{F_2'(x)}{Y} \ln Y +$$
$$\frac{F_3'(x) - F_2'(x)}{Y} + \ldots$$
$$\tau_1 = G_1(x) - \frac{G_2(x)}{Y} \ln Y + \frac{G_3(x)}{Y} + \ldots$$

In addition, from the conditions for matching the pressure function it follows that

$$P_1(x) = p_1(x) = p_e(x), \quad P_2(x) = p_2(x)$$

Thus, relations (3.4)-(3.6) obtained above lead to the establishment of a new law for the behaviour of the flow functions at the outer boundary of the non-linear zone and the inner boundary of the velocity defect region.

In conclusion we note that when the usual boundary conditions at the outer boundary of the turbulent BL are satisfied, in accordance with which (for functions from (3.2))

$$U_1(x, \infty) = \mathrm{T}_n(x, \infty) = \Pi_1(x, \infty) = \Sigma_1(x, \infty) = 0$$
$$(n = 1, 2, 3)$$

on the basis of Eqs.(3.3), it is possible to obtain

$$U_2 = -P_2(x) / U_e(x) + \ldots, \quad U_3 = -{}^1/_2 U_e''(x) y^{*2} -$$
$$p_3^\circ(x) / U_e(x) + \ldots$$
$$V_3 = H_3^\circ(x) + \ldots, \quad P_3 = {}^1/_2 (U_eU_e'' - U_e'^2) y^{*2} +$$
$$p_3^\circ(x) + \ldots \quad (y^* \to \infty)$$

If we now turn to the initial variables x, y, using these expressions and also (3.2) and (3.3), it turns out that in the outer flow region the terms of the second and third approximations (for u, v, p) are of the order of $\varepsilon^2 \ln \varepsilon$ and ε^3. In the case of flow near a flat plate, when $U_e(x) = 1$, $p_e(x) = 0$, the terms of order $\varepsilon^2 \ln \varepsilon$ will not be present, in accordance with (3.8), (3.3), (3.2), and consequently no terms containing $\ln \varepsilon$ will occur in expansions (1.3) or (2.2) either.

Thus, analysis of the turbulent boundary layer on a smooth surface as $R \to \infty$ demonstrates that, contrary to the results of previous investigations /3-6/, it is three-layer. Furthermore the non-linear flow region, situated between the viscous boundary and velocity defect regions, is described by an open set of boundary-layer equations expressing the balance of inertia, pressure and turbulent friction forces. The relative thickness of this region is a value of the order of $\varepsilon^2 = O(1 / \ln^2 R)$. Its existence is not connected with the pressure gradient, although this has a much stronger influence and leads to such important consequences as flow separation in this region (see /9, 10/).

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Translated by C.A.M.

PMM U.S.S.R.,Vol.51,No.4,pp.467-475,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

THE STRONG INJECTION OF GAS INTO A SUPERSONIC FLOW WITH TURBULENT MIXING*

I.I. VIGDOROVICH

A strong distributed injection of gas into a supersonic stream through a permeable plate is examined when the boundary layer (BL) is pressed back from a streamlined surface and the blown gas in the inviscid boundary region is separated from the oncoming stream by a turbulent mixing layer (ML). A disconnection criterion for the turbulent BL on injection and a similarity rule reflecting the fact that the flow over the plate is dependent on conditions at the end of the permeable section are formulated. Universal curves for the pressure distribution and injection-layer depth are given and flow force characteristics are calculated. The applicability of the solution derived from a simpler flow model, in which the ML is replaced by a contact breaking surface, is established, with a corresponding correction for turublent mixing.

1. Formulation of the problem. We examine supersonic flow over a smooth plate positioned at zero angle of attack to the oncoming stream with the injection of gas through a permeable section of its surface. Gas is blown in evenly, perpendicular to the plate with constant flow rate q_w and gas temperature at the wall T_w. Let us assume that, as a result of injection, the BL is pressed back from the entire permeable surface so that the gas blown into the inviscid boundary region *1* (Fig.1) is separated from the outer flow by the turbulent ML which develops from the start of the permeable section. This flow diagram corresponds to experimental data, for example /1, 2/.

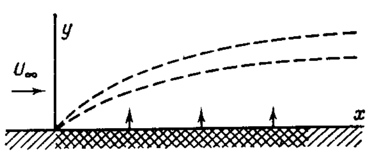

Fig.1

We denote by ε and δ the oreder of relative ML thickness and the inviscid part of the blowing layer. Since the BL is pressed back as a result of the blowing, the transverse component of the flow of mass in the boundary region in order of magnitude is not less than in the ML. The longitudinal component, however, is not greater than in the ML. Hence, and from the continuity equation it follows that $\delta \geqslant O(\varepsilon)$.

Let us examine the non-viscous part of the inflation layer. We shall use dimensionless variables. We will assign Cartesian coordinates to the length l of the permeable section, pressure to P_∞, density to $m_w P_\infty / (kT_w)$, velocity components to $\sqrt{kT_w/m_w}$ and the flow function to $lP_\infty \sqrt{m_w / (kT_w)}$, where k is Boltzmann's constant, and m_w is the molecular weight of the gas blown in; the parameters of the unperturbed oncoming flow are denoted by the subscript ∞. In accordance with the concept of a "thin layer" /2, 3/ we will assume that $\delta \ll 1$. At measured

*Prikl.Matem.Mekhan.,51,4,600-610,1987